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ABSTRACT

Elementary school teacher candidates typically enter their professional training with deficiencies in their conceptual understanding of the topics of elementary school mathematics and with a reliance upon procedural (algorithmic) approaches to the solutions of mathematical problems. If elementary school teacher candidates are expected to teach mathematics in a manner conducive to student conceptual development, then it is critical for the individual candidate to possess personal meanings for the topics of elementary school mathematics and to develop an understanding of the processes involved in student conceptual development. This report describes the pedagogy common to two mathematics methods courses designed to help teacher education students actively restructure their existing mathematical knowledge and expand their views of what understanding mathematics might involve. The teacher candidates are given the opportunity to restructure their own content knowledge using curriculum and instructional approaches similar to those which they will later hopefully use in their own classrooms. Data on mathematical content understandings (rational numbers, fractions, area, volume) and behavioral perceptions (anxiety, confidence, and justification patterns) were gathered from four undergraduate sections and one graduate section of the methods courses. These data indicate significant changes in candidates' mathematical understandings and views of pedagogy. Item analysis of content responses using Student-Problem Curve Theory shows that the candidates' conceptualization of content improves in focus over the course of the class, indicating a deeper level of knowledge structure. Preliminary analysis of structured interviews shows significant shifts in teacher candidates' conceptions of what mathematics is, their view of potential roles in instruction, and methods for evaluating students' success in mathematics relative to their original pre-class positions. A copy of the pretest is included. (37 references) (Author/JJK)

DEVELOPING A PEDAGOGICALLY USEFUL CONTENT KNOWLEDGE IN ELEMENTARY MATHEMATICS

by

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ABSTRACT

If we are to expect elementary teacher candidates to teach mathematics in a way conducive to student conceptual development, we feel it is critical that the teacher candidate possess personal meanings for the topics of elementary mathematics and develop an understanding of the processes involved in student conceptual development. There are often profound instructional implications when this development does not take place. First, the teacher candidate is unable to determine the nature of the conceptual landscape and to recognize and utilize important links between concepts and topics. Furthermore, lacking personal meanings for the mathematics the candidate is unable to be an effective aid in the student's creation of mathematical meaning and cannot gauge when useful and adaptive understandings for mathematical concepts are possessed by the student. In this paper we describe the pedagogy common to two mathematics methods courses (Ed. Studies 408 & 643) designed to help teacher education students actively restructure their existing mathematical knowledge (see Peck & Connell, 1990) and expand their views of what understanding in mathematics might entail (Campione, Brown & Connell, 1989; Connell, 1988; Peck, Jencks & Connell, 1989). The teacher candidate is given the opportunity to restructure their own content knowledge using curriculum and instructional approaches similar to those which they will later hopefully use in their own classrooms. Data on mathematical content understandings (rational number, fractions, area, volume, etc.) and perceptions (anxiety, confidence, justification systems, etc.) were gathered in four sections of Ed. Studies 408 and one section of the graduate version, Ed. Studies 643. These data indicate significant changes in candidate's mathematical understandings and views of pedagogy. Item analysis of content responses using Student-Problem Curve Theory (Sato, 1990; Switzer and Connell, 1990) indicates the candidate's conceptualization of content improves in focus over the course of the class, indicating a deeper level of knowledge structure. Furthermore, preliminary analysis of structured interviews shows significant shifts in teacher candidates conceptions of what mathematics is, their view of potential roles in instruction, and methods for evaluating students success in mathematics relative to their original pre-class positions.

Introduction

Teacher candidates typically enter their professional training deficient in personal meanings for the topics of elementary mathematics which could serve to guide their thinking and suggest ways to them for designing conceptually based curricular experiences for children. Their approaches to mathematical situations are procedural and algorithmic. They perceive problem solving as recalling rules and applying them in terms of word structures rather than the underlying information communicated by the problem situation itself. The teacher candidate is unable to determine the nature of the conceptual landscape and to recognize and utilize important links between concepts.

This paper will describe pre-service teachers entering their mathematics methods training and some consequences of applying conceptual change strategies to their instructional program. The discussion will proceed on the basis of an analysis of pre-training test results and follow-up interviews concerning some basic mathematical concepts, a description of the instructional program, the outcomes of training as regards conceptual changes managed over the ten week period, and some possible implications for teacher training .

As a vehicle for discussing traits of incoming teachers, we will focus on the results of a rational number survey employed to evaluate prospective teachers' understandings of the meanings associated with fraction symbols. We will restrict ourselves to specific descriptions of observations with one class as being typical of the results spread out over the period of two years with both prospective and in-service teachers. Allusions will be made as to the results with the other groups but space will not permit analysis of results with all groups.

The pre-test mathematics course work of the students varied. Of the eighteen represented in this group, one had courses in calculus, six had had college algebra and the balance had recently taken intermediate algebra in preparation for the university graduation requirement of college algebra. In addition, all the students had completed a two quarter sequence in the mathematics of the real numbers which were prerequisite for entry into the elementary teacher education program and the subsequent methods course.

Pre-test Results

A fourteen problem set of rational number exercises were administered to four classes of prospective elementary teachers following their content preparation and at entry to the mathematics methods course required (See Appendix A). The same fourteen problem exercise was also administered to one graduate group consisting of practicing elementary teachers seeking masters degrees. For purposes of discussion, we will focus on one class in terms of specific data. The results are directly extendable to all other groups. The data was organized by arranging the scores

in a matrix that ranked the students from top to bottom in terms of the most right answers (+ = correct, 0 = incorrect), and left to right in terms of the easiest to hardest problems as evidenced by the performance of the group as shown in Table 1.

Student Problem (SP) Chart Analysis of Rational Number Survey

E.St. 408 F '90

Student Number	Test Score (Raw)	(%)	Modified Caution Ind/Sgn	Problem Number 00001010101001 53141826094273
605	11	78.6	0.00	+++++++++++000
602	10	71.4	0.00	++++++++++0000
603	9	64.3	0.06	+++++++0++0000
608	9	64.3	0.06	++++++++00+000
612	9	64.3	0.00	+++++++++00000
614	9	64.3	0.01	++++++++0+0000
616	9	64.3	0.00	+++++++++00000
604	8	57.1	0.13	+++++0+0++0000
606	8	57.1	0.04	+++++++0+00000
617	8	57.1	0.00	++++++++000000
609	7	50.0	0.13	++++0+0+0+0000
615	7	50.0	0.04	+++++0++000000
613	6	42.9	0.00	++++++00000000
601	5	35.7	0.00	+++++000000000
611	5	35.7	0.17	++++000000+000
610	3	21.4	0.06	+000++00000000
618	3	21.4	0.00	++00+000000000
607	2	14.3	0.30	00000+00000000
Score Ranking ↓ Problem Difficulty by Performance →				00001010101001 53141826094273 Problem Number

Table 1.

The arrangement of the scores in terms of difficulty provides information that suggests those to be interviewed. For instance, student #611 in Table 1 missed some of the problems which were for the group relatively easy, but has correctly completed problem 14. This problem was missed by everyone except two students near the top end of the group. This observation indicated something unusual might be taking place and suggests a follow up interview to determine what might be taking place. The fact that problems 02, 07 and 13 were missed by everyone is disturbing. What is there about these problems that causes them to be missed by everyone? An interview with

students from the top, middle and lower ranges of scores might yield some information concerning generally held misconceptions from students of each scoring category.

The modified caution index (MCI) is an indication of the degree to which the students' responses deviate from those expected from a student of this rank within this group. A MCI of 0.00 indicates the student is moving from the easiest to the most difficult without any abnormalities in response occurring. Student #611 has an MCI of 0.17 which indicates that in terms of the score received is missing or getting right some unexpected problems. A high MCI indicates an unusual pattern of responses while a low MCI indicates a more normal response pattern with respect to the groups performance (Harnish & Linn, 1981).

Consider problem 02 which was not correctly handled by any of the students in this group:

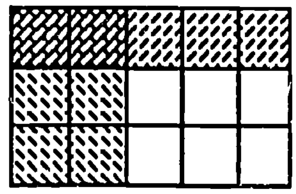
> Mary has socks in two drawers of her dresser. In the top drawer, one third of the socks are white. In the bottom drawer, two fifths of the socks are white. What portion of Mary's socks are white.

The responses to this problem included the following:

11/15 -16 students | 19/15 - 1 student | 3/5- 1 student

The answer of 19/15 was completed by the student (#615-Table 1) somehow confusing what the problem was asking for and writing down the complementary fractions 2/3 and 3/5. Once this was done, the usual algorithm that applies to part-whole addition of fractions was called up and used to get the total of 19/15. This answer means that there are more white socks than there were socks to begin with. Scary isn't it?!! The question of what the answer meant was not a concern to the student since the rules had been applied as the student perceived they should be.

The student (#612 who ranked fifth in the class on this instrument - see Table 1) giving 3/5 as an answer did it by constructing a sketch as follows:



The student had cut the rectangle and shaded in two fifths in the vertical direction and one third in the horizontal direction. In counting the squares, the student counted the six squares that correspond to 2/5, but when determining the thirds the student indicated two of the squares (the double shaded ones in the sketch) had already been counted so she only added in the remaining three getting a total of nine and wrote 9/15. This result led, via reducing, to 3/5. The follow-up interview indicated that this student was also looking at the problem in terms of the part-whole concept of fractions and did not recognize the inappropriateness of the response. The student was responding to a model she had been exposed to in a previous course. The model was being used as a mnemonic, that is, as a means of remembering and not as a guide for thinking.

The answer 11/15 represented the consistent response of almost all those who took the test, both practicing teachers and pre-service teachers (See appendix A). By way of gaining some insight into the thinking typical of those who responded in this manner, consider the following interview. It should be noticed that, in terms of content preparation as measured by course work, this student (#605-Table1) was the best prepared in the class having taken calculus prior to the mathematics sequence for elementary teachers.

> Interviewer: Can you show me how you did this one?
>
> Student: Well you have to add so first I had to find a common denominator. That's fifteen. One third is five fifteenths, and two fifth is six fifteens, so that 's eleven fifteenths altogether.
>
> Interviewer: Does this answer seem reasonable to you?
>
> Student: Sure. See I did it like this.. (Student then proceeded to repeat the algorithmic arguments used originally) I know that I'm right because I get the same answer two times in a row.
>
> Interviewer: How did you know you needed to add in the first place?
>
> Student: It wants to know what it is altogether.

Interviewer: Tell me, are more or less than half of the socks in the top drawer white?

Student: Less.

Interviewer: What about the second drawer? Are more or less than half of the socks white?

Student: Less.

Interviewer: What about the combined drawers, are more or less than half of them white?

Student: "More ...er... hmm...(long pause) Something's wrong.

Interviewer: Oh! What's that?

Student: It doesn't seem right.

Interviewer: What doesn't ?

Student: Well it seems that less than half ought to be white.

Interviewer: Well what do you think now?

Student: Well, I'm not really sure.... it doesn't seem quite right.... but I know that I did the problem right. That's the way we were taught in ... (mentions prior math content course and instructor).

Interviewer: I see.

Another student when confronted with the dilemma described above in the course of an interview, worked through the problem using paper pieces to represent the socks, three pieces for the top drawer and five pieces for the second drawer and marking them to represent the white and non-white socks. When struck with the conflict between her answer of 11/15 based upon her application of the rules and her answer of 3/8 from actually attempting to "figure this thing out" stated, "Well I would like to believe that this (3/8) is right, but I'm afraid that my rules just won't let me."

The interviews for this group suggested that each student had observed the language involved in the problem statement, determined that addition was the required operation, and launched into the standard algorithm without regard as to whether or not the rule applied or their answer was reasonable. This indicates that the meanings these student's had were procedural in nature and not

amenable to sense-making or reality checking. It is disturbing to note that these observations held true for all of the students in this group. Even when confronted with the fact that if their answer was correct, the socks must have been multiplying like rabbits right while they were working on the problem, they expressed a faith in the rules, often in opposition to the reality that faced them.

Observed Perceptions

As the rational number study progressed it was extended to include in-service teachers as well. When this was done (see Table X in Appendix B) a varied, but individually consistent, pattern of beliefs was detected. Among those beliefs, the following were common:

1) Mathematics is computation. The computational form is of critical importance.

2) Mathematical problems should be quickly solvable in just a few steps.

3) The goal of doing mathematics is to obtain "right answers".

4) Patterns are sufficient evidence for accepting a rule. "If it works a few times, it works all the time."

4) Their role as a mathematics student is to passively receive mathematical knowledge from experts and to demonstrate that it has been received usually by returning it as close to form as they can remember on paper and pencil tests.

5) Solving problems consists of recalling and applying specific algorithmic rules that relate to specific kinds of problems.

These beliefs correspond closely to those commonly exhibited by public school students (Frank 1988) and stand as distinct barriers to the desired higher level goals of independent thinking and problem solving (Peck 1981, Cobb 1985). In short, these prospective elementary teachers lack meanings about rational numbers sufficient to guide their thinking or enable them to solve problems other than specific cases tied to an algorithmic approach. Their faith in themselves is not sufficiently strong- they are dependent upon the thinking of others - not their own.

Instructional Program

In terms of the goals set out by professional groups interested in mathematics education, changes are required in the scope and the implementation of teaching strategies to focus on mathematical thinking and problem solving (NCTM, 1990; etc.). The meanings, understandings, perceptions and beliefs about mathematics described above, in general effectively block the envisioned progress and growth of teachers toward these goals. Teachers unable to find meaning in mathematics lack insights essential to guiding young students into the process themselves. If we are to help, we must break into the rote memory emphasis of instruction, preoccupations with formalisms and unproductive belief systems by bri g meaning into the training of elementary teachers. Henderson (1987, p. 236) points out that meaning comes from building intuition:

> In the schools today formal geometry (with its postulates, definitions, theorems and proofs) is usually considered to be the apex or goal of learning geometry. Informal geometric topics and activities which do not fit into the formal structures are often given second class status and relegated to the domain of mere motivation or help for those who are not smart enough to learn the "real thing"-- formal geometry. I am a mathematician and as a mathematician I wish to argue that this so-called informal geometry is closer to true mathematics than is formal geometry. I do not believe that formal structures are the apex or goal of learning mathematics. Rather, I believe, the goal is understanding -- a seeing and construction of meaning. Formal structures are powerful tools in mathematics, but they are not the goal. I don't blame teacher for giving formal geometry too much emphasis; mostly I blame my fellow mathematicians because we have done much to perpetuate the rumor that formal systems are an adequate description of the goal of mathematics. The mathematician will use formal systems to help in the explorations but the driving force and motivation and ultimate meaning come from outside the system.

The systematic organization of learning experiences into formal structures, such as Henderson has alluded to, tends to three flaws (Szilak 1976): First teachers are placed in the position of equating originality with imitation; that is, the repetition of formal arguments as evidence of mastery and original thinking. Second, learning is reduced to rote, which has a tendency to fragment and compartmentalize learning and thus interfere with further understanding. As a result, the interconnectedness between ideas is sacrificed and students build up isolated bits of

knowledge that Whitehead (1916) has referred to as "inert" and as such do not result in a usable set of tools for problem solving. Finally, learners, when measured by the usual tests, give evidence they have "mastered" knowledge that they do not really comprehend. The evidence outlined above lends credence to these "flaws" and they would seem to be fatal deterrents to envisioned progress in our educational endeavors. The evidence also suggests that higher education may exacerbate the problem by its focus on abstract formalism and structure of the subject thus deepening the ties to rules and procedures as the ultimate aim of instruction (Shulman 1987).

These comments reflect the observations and outcomes of interaction with the groups described earlier and form a basis for the modifications attempted over the quarter of instruction. The balance of this paper describes an approach designed to help these prospective teachers set aside some of their counterproductive perceptions and strategies and replace them with a focus on meanings.

The methods course was founded upon four beliefs:

1. Mathematics is a way of thinking.

2. The goal of instruction is mathematical thinking and problem solving.

3. An overriding goal is to provide a foundation upon which powerful mathematical generalizations can be constructed.

4. Learners must be actively involved in the construction of their own knowledge base.

The alternative to teaching for memory is to give students a base from which to build meanings, to reason out answers, and to verify the many decisions they need to make before they can be sure of their own thinking. The fundamental axioms for the real number system provides an inadequate base for elementary school children and their teachers to reason from because the axioms are verbal generalizations removed from their experience. As such, they are not "fundamental" to a beginner. Physical materials provide an alternative and are used in the course for the express purpose of building personal meanings on the part of the prospective teacher and establishing a base upon which abstractions can be safely built.

How physical materials are used is of critical importance. It is well known that exposure to manipulatives does not necessarily result in the use of the materials as guides to reason (Davis 1980, Holt 1982, Baroody 1989.) It is important to use the materials in connection with problem situations in order to assure that the mental referents developed become tools for decision making. Problems are posed which form a basis for the concepts at hand, and questions are asked which require the students to use the materials to solve the problems and make decisions (Jencks 1987). No old remembered rules allowed.

A word must be said about the role of the instructor and the expected role of the students. The instructor and student reverse traditional roles. Realization of the goals of the course is impossible if the course is taught by drill, by lecture, by telling students how to do the problems, or by other popular methods which insulate the students from complete immersion in conceptualizing for themselves. It is necessary for them to solve problems daily and determine for themselves whether they have solved them sensibly. Thus, the instructor becomes a problem poser and question asker. Students solve the problems and explain. No answers are given by the instructor, and no answer book is available. Using physical materials from which to reason, students are expected to find ways to confirm their own decisions about the sensibility of their results and defend them to their peers and the instructor. They work in small groups of up to four and share their thinking and reasoning.

As an example of a problem posed near the beginning of the course, consider the following one centering upon place value and the operation of division as an example before we go on to describe the progress and gains over the ten week period (Class meets daily for 50 minutes):

The students had been involved in constructing a base five place value system using multi-base arithmetic blocks as shown in Figure 1 as a referent system for candy packaging (Wirtz, 1967; Jencks 1987).

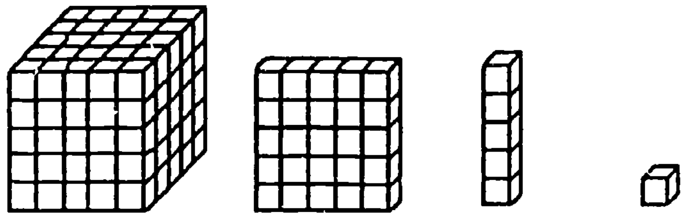


Figure 1.

> The instructor posed the problem: "I noticed that as you were sharing some of these arrangements between two people that sometimes you had a left over and sometimes you didn't. If there is a left over we say the number is odd. If there is no left over, the number is even. Is there some way you can determine, without dividing, whether or not a number in the five packaging scheme is odd or even?"

After some experimentation, one student proffered the following to her group and the instructor:

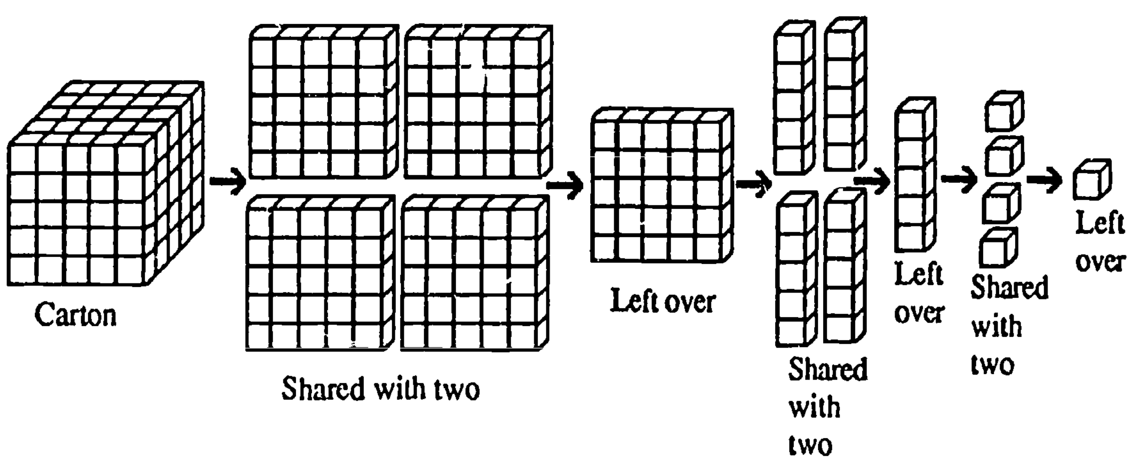
> Student: I found a quick way to decide was to just add up the digits. If the result is odd then the number is odd. If it's even, then the number is even.
>
> Instructor: I don't see how you can do that. You have the big carton and you want to add it to that flat? If you do, you get two what's? It doesn't seem right to me that you add all these things together and then say that lets you say something about a single block being left over. Do any of you understand how that happens?
>
> Another Student: I've checked it. It seems to work. It's a pattern.

The other students checked a few problems out as well and agreed that this method worked for all the numbers they had tried even though some of them had other patterns which also seemed to work. None of them, however, could provide a logical explanation for why adding digits resulted in permitting them to say a single block would be left over if the number was odd. They were willing to accept unjustified patterns in place of solid evidence. It was not easy for them to understand why the instructor would not accept such superficial evidence and why he insisted that the rationale for deciding was more important than the answer.

Instructor: I still can't see how that would work. You would be adding packages together of different sizes. To share 3231_{five} with two, you would be adding 3 cartons, 2 flats, 3 rods and one single together. That is, $3 + 2 + 3 + 1 = 9$, but 9 what? I can see adding 3 cartons to 2 cartons, or 2 flats to 3 flats, but I can't see adding cartons to flats. I don't believe that method will work all the time. It may for these small numbers, but certainly not for all numbers in the five packaging system.

The problem of showing why this worked or didn't work was left for the students to work on overnight. They were assigned to develop some clear arguments as to why this method or any other methods they had developed would necessarily have to work. The following is an argument that was developed by one of the students:

Student (picking up a carton as shown in the diagram below): To share this carton with two people means you would have to open it getting five flats. When you do it (share the five flats) with two, there is one flat left over. Sharing the flat (between two) you (have to) open it getting five rods and there is one (rod) left over. If you open the rod to get five singles and divide by two you get one (single) left over.



The student repeated the argument for a flat and a rod and summarized the argument as follows:

Student: To decide whether 3231 (five) is odd then there are 3 leftovers from the cartons (one from each carton), 2 leftovers from the flats (one from each flat), three from the rods (one from each rod). When you add $3 + 2 + 3 + 1$ you are only adding up the singles. It adds to 9 (singles). When these are shared with two, there is one (single) left over, so the number is odd.

This student's explanation was discussed until the class understood why the method of adding the digits to get at the question of odd or even worked. Other methods were dealt with in similar ways.

The argument outlined above represents an expanded gateway to thinking about numeration systems and divisibility which is not available to those who simply learn the rules for dividing or to identify place value positions by name. It illustrated a move away from dependency on rules or instructor explanations and that the students were thinking for themselves based upon the meanings developed from their experiences with the physical model. Only one person seemed unable to defeat unproductive perceptions and was constantly looking to the manipulations of symbols as the key to progress. The balance of the students began to lose some of the counterproductive beliefs they had brought to the class. Even more important, some of the students began to explore self-generated questions about what might be true in other bases or might be true only of even-numbered bases or just odd-numbered bases. Stimulated by these kinds of questions, the students came up with insights and meanings which demonstrated a greatly expanded view of numeration systems. The meanings and insights gained were extended to expand their understandings beyond whole number operations to include fractions, decimals and some basic aspects of geometry. Later in the course, the meanings acquired became the basis for solving problems like the following:

> Instructor: I wonder if there is a fraction from which .1212121212... came from?

The students had investigated fractions and place value and extended those meanings to decimals and were able to convert common fractions to decimals by appealing to the objects. For instance, to express 2/5 as a decimal the students referred to some base 10 multi-base arithmetic blocks as shown in Figure 2 and generally argued as follows:

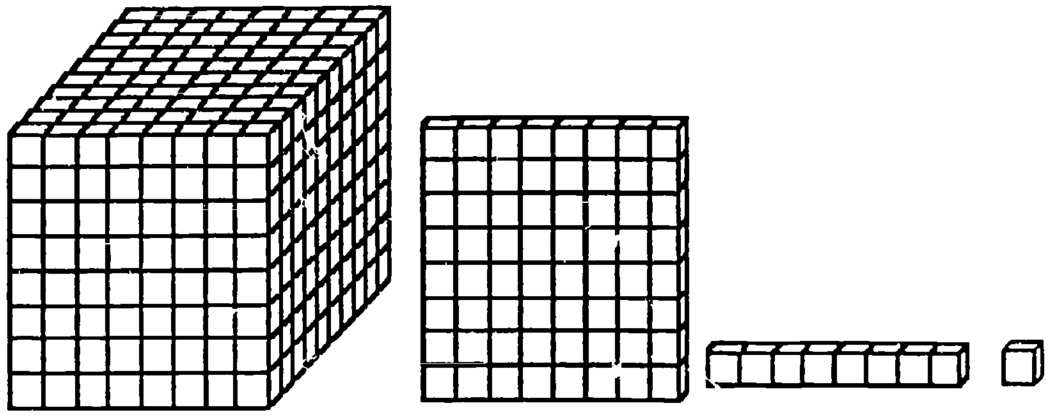


Figure 2

> If I unpackage the cube, I get ten flats. Each one is a tenth. I share them with five and each one gets two tenths. Two shares would be four tenths, so 2/5 = .4.

With this meaning in place, the students explored and found that many fractions yielded repeating decimals of infinite magnitude. For example, 2/11 was converted to a decimal as follows:

> Student: I break the cube up into tenths, no one gets any so there are zero tenths. (wrote .0) So I break them up into long rods (hundredths), now there are a hundred of these. Each one would get 9 with one left over (wrote .09). The one rod left over is broken into thousandths. No one can get one (wrote .090) so I break those down into ten thousandths (demonstrated a cutting action on a thousandths cube). There are one hundred of these, so each person gets 9 (ten thousandths) and there is one left over (wrote .0909). This will break down into ten (100 thousandths), so there will be a zero. Then breaking them down further, a 9 (millionths) and it will continue forever. This would be one share (1/11) so two of them (2/11) would be .18181818... .

The students did a number of fraction to decimal conversions and found some interesting patterns which they explored and explained via the place value model how and why the patterns occurred. The instructor then posed the question stated above:

> I wonder if there is a fraction from which .1212121212... came from?

The students went to work. In the initial stages they made some estimates and engaged in some trial and error efforts. Then some ideas began to be developed which helped narrow the

search. One student suggested if they could find .06060... then that would help because .12121212... would just be two of them. From here various students suggested that .01010101..., .020202...,.0303030..., .04040404..., would also work since .121212... was just a multiple of any one of these. One student suggested that if they were looking for .01010101..., the share number (denominator) must be larger than ten because, ". . . if it was smaller then each person would get a tenth and you would have to fill in the tenths place." The problem was left overnight.

The next day most of the students (three exceptions) said they could give a fraction for any repeating decimal. The instructor asked them to hold their arguments and explanations and just give answers to some repeating decimals he posed. They were able to give answers and they were checked out (by benefit of a calculator). For the original posed problem (.121212...) the class agreed that 12/99 or 4/33 would be the fraction from which it was generated. Asked how they came upon the solution, several students suggested different approaches. The following was representative:

> Student: J suggested yesterday that the share number (denominator) had to be bigger than ten to get .01010101... I figured if I could get .010101.. then I could get fractions for any other decimal that repeated in two's.
>
> Classmate: How do you know that.
>
> Student: Cause any repeating decimal in two's (which repeats in blocks of two) is just something times .01010101... . For instance, .565656... would just be 56 x .01010101... (She provided an argument showing the exchanges in terms of the place value model.) Since the share number had to be greater than ten, I got to fooling around to see what might work. I found soon that the number would have to be such that the left over in sharing the hundredths would have to be large enough that there could not be a number in the thousandths place either when they were shared out. I chose 99 and found that there would be ten one-thousandths which would open to 100 ten-thousandths,then each one would get one ten thousandth and there would be one (ten thousandth) left over, which opens to ten (hundred thousandths) then 100 (millionths) and there would be one (miilionth) left over. The constant left over of one would break down to one hundred in two steps and then can be divided by 99 with one left over and the process would start over again. So all I need is 12/99 to get .12121212... .

> If I need to do a three (digit) repeating one (decimal fraction), then I would look at this (writing down .001001001001...). Then 999 would go into the thousandths with one left over. It would take three exchanges (to the millionths place before it would go in once again. It breaks down to tens, hundreds, thousands in three steps getting the two zero's in there. So any decimal that repeats in three's would be this (one of the repeating blocks) over 999.

The students discussed this and other variations until they were clear on what happened. They also showed how to extend their arguments to three digit repeating decimals, four digit repeating decimals, etc. The instructor then asked how they would deal with a decimal that started out with a non-repeating block and then repeated like .35121212... Overnight they had the problem solved and could say with personal conviction that any repeating decimal had a common fraction equivalent. The meanings these students had acquired for place value and common fraction had been drawn upon to provide a solution. This represents a decided shift toward the declarative mode of thinking.

Observed Outcomes

This research was performed and implemented in a natural setting consisting of existing physical surroundings and relationships to ongoing programs and was influenced by them. As a consequence, real world events often dictated the actual conduct of the study. Because of this, it is important to remember that the research reported, as is the case in all research settings, is the result of and reflects some compromise.

Because the population of students in the elementary program was small, each class was the only one involved in any particular quarter. As a consequence, the information gathered is without contrast to controls. The period of instruction was short and the numbers of students involved in any one group was small. In addition, there is no mechanism in place for continuing to monitor our graduates as regards the actual impact or permanency of change or if there is permanent change that affects actual classroom differences in instructional practices. Because of these factors, a case

study methodology was used to the report the findings. Despite reliance upon case methods, ongoing use of SP charts and interviews were used to buttress the case method.

The concerns outlined in the previous paragraph will be addressed via a large scale effort involving various elements of the faculty and the public schools which is being mounted in the fall of 1991. This research effort will provide proper controls, evaluation points and follow through as to results.

The ability to perform on problems similar to those used in the original rational number survey was noted from performance on in class problem sets given during the course of instruction. All eighteen of the students ranked high and errors were of the inadvertent variety with again the one exception. The shift in responses on familiar problems would not be meaningful anyway unless scores went down, which they never have. We would be guilty of testing the students on similar problems and in Schoenfeld's (1982 p.29) words "...patting ourselves on the back ..." while not really knowing if there was genuine progress toward mathematical thinking and problem solving. We choose instead to examine the students responses to an examination problem that required them to link their understandings to a unique problem setting. The students learnings concerning place value and the meanings they had acquired permitted them to effectively and thoughtfully deal with what are often memorized relationships between common and decimal fractions and a sense of infinite sequences not always present even among calculus students.

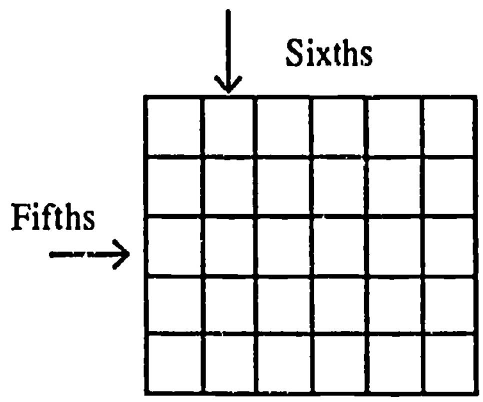
This class was given a final examination which included three aspects. The first two had to do with understandings related to developed models underlying the common elements of elementary arithmetic. The third represented a genuine problem derived from one of the areas investigated during the quarter. The problem did not allow them to simply recall previous results, but required them to deeply investigate in order to solve. Since problems of this type typically require considerable effort on the part of the students, only one is given to avoid a time pressure constraint which might cause them to attempt a reversion to memorized rules (Connell & Peck, 1991).

These problems are also designed so that a broad generalization can be derived from them and extend the students understandings of the mathematics even a bit deeper. This makes it a learning experience in and of itself. Each quarter a different problem is given to prevent a sharing of experiences prior to the evaluation.

The following problem was given to this class as an opportunity for them to demonstrate that they could independently solve an unfamiliar problem and defend themselves:

> Demonstrate you can use the fundamental meanings associated with the part whole concept of fraction to solve an unfamiliar problem by exploring the following:
>
> > Ancient Egyptians, according to interpreters of the Rhind Papyrus, were able to use fractions in the common business of their civilization. Two oddities, however, were noted. First, they could only use fractions with one in the numerator, so they would chain these unit fractions together to represent a given amount. Second, in these chains of fractions they never repeated one of them, a given fraction would only appear once. For example, if the Egyptians wanted to express an amount equal to 2/7 they might write something like 1/7 + 1/8 + 1/56 (Is this really equivalent to 2/7? Better verify it.). As a variation on the Egyptians thinking would you examine the following set of fraction equalities:
> >
> > $$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$
> >
> > $$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$
> >
> > $$\frac{1}{4} = \frac{1}{5} + \frac{1}{20}$$
> >
> > Verify whether the above are true or not. If so, is it possible that any given unit fraction (a fraction with one as the numerator) can be expressed as the sum of two unit fractions like those above (remember no fraction can be repeated in the sum)? If there is a way to do this, explain, using an appropriate model and no rules, why it is always possible to write any given unit fraction as the sum of two other unit fractions. If there is an exception which shows it can't happen, please list it and show why it won't work?

The arguments put forth by the students pointed to significant growth toward independent thinking and problem solving. They argued that it was indeed possible to express any given unit fraction as the sum of two different unit fractions where one of the sum had a denominator just one bigger and the other was the product of the original fraction and the one with the denominator just one larger (i.e. $1/n = 1/(n+1) + 1/n(n+1)$). A typical explanation involved sketching a rectangle and cutting it as shown below:



In this instance, the student cut the rectangle into fifths one way and sixths the other and used this model as a basis for arguing the general case as follows:

> Student: Each fifth (horizontal strip of squares) has one more piece (square) than each of the sixths (vertical strip of squares). So to make them equal (1/5 to 1/6) you need to add one square to it (1/6). There are 5x6 or 30 squares so one square is 1/30 (of the rectangle), so 1/5 (six squares) equals 1/6 (five squares) plus 1/30 (one square). ... It works all the time because each row will have one more square than each strip down, so to make them equal all you have to do is add in one square which is times them (the denominators) both. It's one over the number of squares which is times them (the denominators) both.

In short, the student is saying (albeit in very poor English) that $1/n = 1/(n+1) + 1/n(n+1)$ is always true because each 1/nth of a rectangle n by n+1 will have just one more square than each 1/(n+1)th, and that one square is represented by 1/n(n+1)th of the rectangle.

As the students are examined via such problems, they reveal growth toward an ability to utilize the meanings and concepts mastered in previous experience and link them to other more complicated situations. Of course, the amount of growth and the depth of understanding varied

from student to student and there is not a good way of distinguishing them from each other. Many of the students simply express themselves better than others in written and verbal communication. Leaps in conceptual understanding occurred at various stages and some students took longer than the others to break out of their built in perceptions to deal with the limited amount of material that can be presented in a short ten weeks. Yet, all but one of this group made significant strides toward attitudes and perceptions that are considered conducive to independent thinking and problem solving. They demonstrated their growth via solving and explaining significantly difficult (for them) problems. More important than the fact that they were able to do this are the following observations which with small numbers of exceptions (1-3 in each class) are true of the students involved:

1. They expected to solve the problems. They were not waiting for an explanation or a formula.
2. They were not willing to accept a simple pattern as evidence, but expected to show how and why the pattern worked.
3. They were willing to work on a problem for extended periods of time without giving up.
4. They went beyond the simple production of an answer to develop and defend general methods of solution.
5. They did not expect the instructor to show them how, but were willing to work it out themselves.

Results with students in the other four classes and in the graduate class comprised of practicing teachers were remarkably similar. As the reader may find in the SP charts for these students and the practicing teachers, there is very little difference in the outward performance of the groups. Only occasionally does one find a student that has a grasp of essential meanings associated with the part-whole meaning assigned to rational numbers. By the end of the quarter the students showing significant advances toward becoming better thinkers and problem solvers themselves and begin to envision ways they can interact with children to involve them in similar concerns.

Some Concluding Comments

The need to improve educational programs has been amply documented in the literature. Comparisons with other countries (NAEP 1987, Kroll 1987, pp. 36-43), The large numbers of foreign students occupying graduate level spaces in universities (Report of the Ad Hoc Committee on Resources for the Mathematical Sciences, 1984 p. 53), and the research data describing the magnitude of misconceptions and the inabilities of our graduates to address problem situations (Gentile, 1986, pp. 159-178) are only a small sampling of evidences that can no longer be ignored. We must address the problem or find ourselves and our children increasingly unable to deal with rapidly evolving world society we find ourselves a part of (Kaput, 1986). The evidence we have described above points to an education system that is preoccupied with the appearance of success via a primary emphasis on easily measured manipulative skills (Schoenfeld, 1982, p. 29). The desirable and necessary meanings and understandings requisite to rational application of principles in thinking and problem solving are somehow left out. Rosnick(1980, p. 35) summarizes the problem in the case of mathematics:

> Several members of our research group are finding, in pilot studies, that students misconceptions are not limited to the reversal of equations, but that there are a number of other deep seated misconceptions...That many students can succeed in a curriculum to the point of becoming engineering and science students *and teachers* (Italics added), yet somehow have missed the mathematically essential notions of equation and/or variable is disturbing... It suggests that an even larger proportion of non-science students are not gaining the skills ... and are slipping through their education with good grades and little learning.

We may not want what we are getting as a consequence of the educational experiences of our children, but we are getting what it is we are asking for. If we want something different, we must have the courage and determination to change the questions. In particular, it is the considered opinion of the writers that we must redirect the emphasis on the formal structure of knowledge and the means of evaluating success toward the acquisition of meanings which are accessible to teachers and their students, and which form the basis upon which the powerful formal structures of

mathematics can ultimately be constructed. This will not be easy and we may stumble often, but to fail to do so will cause us to continue to construct edifices of knowledge that, for many students, crumble for the lack of an adequate foundation.

We have tried to point out in this paper one small attempt to set the stage for constructing lasting long term changes in the mathematics education of our teachers. We have described some pilot studies that focus on engendering meaning and helping teachers construct adequate belief systems and through them hopefully their students. Efforts like these must be expanded and better tools found for evaluating results which include beliefs and perceptions that encourage or discourage positive movements towards mathematics. It is the belief of the authors that should we fail in this aspect of bringing reform, other plans for improvement such as increased time on task, more testing, increased course requirements for prospective teachers, and even more money, will not make much difference.

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Appendix A

RNS

Date____________

Name____________________________________ Instructor_________

Do as many of the problems below as you can. Leave out any that you feel uncom with. Do your work in the space provided and write your answer in the appropria

1. Which fraction represents more? $\frac{3}{7}$ or $\frac{4}{9}$ 1.______

2. Mary has socks in two drawers of her dresser. In the top drawer, one third of the socks are white. In the bottom drawer, two fifths of the socks are white. What portion of Mary's socks are white? 2.______

3. $3\frac{1}{3} - 2\frac{1}{2} =$ 3.______

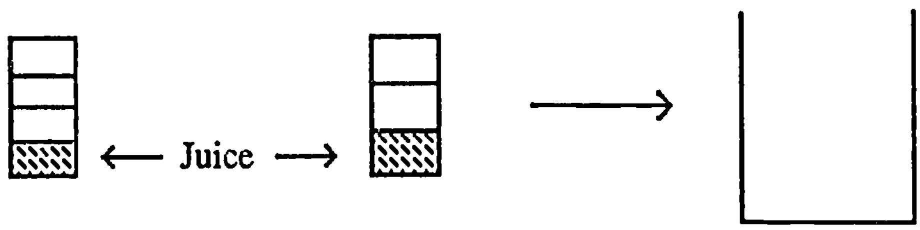
4. $2.01 \times 1.01 =$ 4.______

5. Place a number in the circle to make the statement true? $\frac{4}{6} = \frac{\bigcirc}{18}$ 5.______

6. $2\frac{2}{5} \times 4\frac{3}{7}$ 6.________

7.________

7. Two glasses of equal volume are filled with a mixture of juice and water. The first glass is 1/4 juice and the second is 1/3 juice. If the two glasses are combined by pouring them into a large container what portion of the mixture in the large container will be juice?



8. $3\frac{2}{5} \div 2\frac{3}{7}$ 8.________

9. Place a number in the circle to make the statement true: 9.________

$$\frac{\bigcirc}{4} = \frac{2}{5}$$

10.________

10. The little Red Hen baked two loaves of bread. She decided to share her efforts with her barnyard friends. If she retained 2/3 loaf for herself, gave the dog 1/2 of a loaf and the chicken 2/5 of a loaf, how much was left for the pig?

11. A carpet layer earns $0.73 a yard for laying carpet. If he layed 44.79 yards one afternoon, how much did he earn? 11.______

12. Suppose the carpet layer mentioned in problem twelve above earned $55.79 one day, how much carpet did he lay that day? 12.______

13. Two bikers start from the same point and the same time and travel in opposite directions. At the end of 20 minutes, they are 11 miles apart. If one of the bikers is going 1 1/5 times as fast the other can you determine how fast each is going? 13.______

14. In a certain dessert the ratio of sugar to flour is two to nine. How much sugar is needed for a mixture containing thirty three cups of flour? 14.______

Appendix B

Student Problem (SP) Chart Analysis of Rational Number Survey

Math. 406. Su. '90

Student Number	Test Score (Raw)	Test Score (%)	Modified Caution Ind/Sgn		Problem Number 00001100101001 53411268094273
304	11	78.6	0.00	A	++++++++++0000
305	11	78.6	0.34	B	++++0++0++++00
302	10	71.4	0.00	A	++++++++++0000
303	10	71.4	0.09	A	+++++0++++0000
313	10	71.4	0.28	B	+++++++00+0++0
319	10	71.4	0.00	A	++++++++++0000
301	9	64.2	0.14	A	++++0+++0+0000
314	9	64.2	0.11	A	+0++++++++0000
316	9	64.2	0.00	A	+++++++++00000
306	8	57.1	0.07	A	++++++00+00000
315	8	57.1	0.07	A	++++000+++0000
318	8	57.1	0.09	A	+++++0+0+00000
308	7	50.0	0.10	C	+0+++0+++00000
307	6	42.9	0.07	C	+++00+++000000
317	6	42.9	0.15	C	+++00+0+000000
320	6	42.9	0.00	C	+++++000000000
322	6	42.9	0.10	C	++0++00++00000
312	5	35.7	0.18	C	+++0+00000+000
310	4	29.0	0.04	C	+++000+0000000
309	3	21.4	0.09	C	+00++000000000
311	2	14.3	0.11	C	+0000+00000000
321	2	14.3	0.18	C	++000+00000000

Score Ranking ↓

Problem Difficulty by Performance →

00001100101001
53411268094273
Problem Number

Table 2

Student Problem (SP) Chart Analysis of Rational Number Survey

E.St. 408. S '90 Student Number	Test Score (Raw)	(%)	Modified Caution Ind/Sgn		Problem Number 00001010101001 51341806294723
424	12	83.3	0.05	A	+++++++++0++00
422	10	71.4	0.00	A	++++++++++0000
411	8	57.7	0.10	A	++++++0+00+000
418	8	57.7	0.00	A	++++++++000000
409	7	50.0	0.00	C	+++++++0000000
414	7	50.0	0.14	C	+0++++0++00000
417	7	50.0	0.13	C	+++++000++0000
420	7	50.0	0.00	C	+++++++0000000
421	7	50.0	0.18	C	+++0+0++0+0000
403	6	42.9	0.04	C	++++0++0000000
410	6	42.9	0.11	C	++0++0+0+00000
412	6	42.9	0.07	C	++++0+0+000000
413	6	42.9	0.11	C	+0++++00+00000
416	6	42.9	0.11	C	+0++++00+00000
419	6	42.9	0.15	C	+0++0++0+00000
402	5	35.7	0.07	C	++0++0+0000000
404	5	35.7	0.07	C	+++0+0+0000000
407	5	35.7	0.07	C	+++0+0+0000000
408	5	35.7	0.16	C	++00++0+000000
415	5	35.7	0.32	D	++0+00000+0000
405	4	29.0	0.00	C	++++0000000000
401	3	21.4	0.07	C	00+++000000000
406	3	21.4	0.16	C	++00000+000000

Score Ranking ↓ 00001010101001
51341806294723
Problem Difficulty by Performance → Problem Number

Table 3

33

Student Problem (SP) Chart Analysis of Rational Number Survey

E.St. 408. Su '90

Student Number	Test Score (Raw)	Test Score (%)	Modified Caution Ind/Sgn		Problem Number 00010101010010 51413284908732
003	12	85.7	0.00	A	++++++++++++00
005	12	85.7	0.10	A	+++++++++++++0
002	11	78.6	0.00	A	+++++++++++000
011	11	78.6	0.00	A	+++++++++++000
007	8	57.1	0.30	B	+++++0+00000++
001	7	50.0	0.02	A	++++++0+000000
004	6	42.9	0.10	C	++++0++00+0000
012	6	42.9	0.02	C	+++++0+0000000
006	5	35.7	0.00	C	+++++000000000
009	5	35.7	0.11	C	++0+++0+000000
010	5	35.7	0.00	C	+++++000000000
008	2	29.0	0.00		+0+00000000000

Score Ranking ↓ | 00010101010010
Problem Difficulty by Performance → | 51413284908732
Problem Number

Table 4

34